THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010D Advanced Calculus 2019-2020

Problem Set 4

1. Without using any software, sketch the graph of the following functions.

   (a) $f(x, y) = x^2 + y^2$

   (b) $f(x, y) = x^2 - y^2$

   (c) $f(x, y) = -x^2 - y^2$

   For each of the above function, determine whether $(0, 0)$ is a maximum or minimum point.

2. Let $f(x, y) = \sin(x^2 + y^2)$.

   (a) Plot the graph of the function $f(x, y)$.

   (b) Describe the level set $L_{-1}(f)$, $L_0(f)$ and $L_1(f)$.

3. Let $f(x, y) = e^{-x^2-y^2}$.

   (a) Plot the graph of the function $f(x, y)$.

   (b) Describe the level set $L_c(f)$.

4. Let $f(x, y) = \begin{cases} 1 & \text{if } |x| = |y|; \\ 0 & \text{otherwise.} \end{cases}$

   (a) Sketch the graph of the function $f(x, y)$.

   (b) Prove that $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

5. Let $f(x, y) = \dfrac{xy^2 - 1}{y - 1}$. Prove that $\lim\limits_{(x,y)\to(1,1)} f(x, y)$ does not exist.

6. Determine whether each the following limit exists, if yes, find its value; if no, prove your assertion.

   (a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 + x^2y + 3xy^2 + 3y^3}{x^2 + 3y^2}$

   (b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 + \sin^2 y}{2x^2 + y^2}$

   (c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^3}{x^2 + y^2}$

7. Determine whether each the following limit exists, if yes, find its value; if no, prove your assertion.

   (a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

   (b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$;

   (c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^3 + y^4}$;

   (d) $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy^2 - 1}{y - 1}$.

8. Let $f(x, y) = \dfrac{xy^3}{x^3 + y^5}$.

(a) i. Let $\gamma(t) = (t, mt)$, for $m \in \mathbb{R}$. Show that $\lim_{t\to 0} f(\gamma(t)) = 0$.

ii. Let $\gamma(t) = (0, t)$. Show that $\lim_{t\to 0} f(\gamma(t)) = 0$.

(b) Let $\gamma(t) = (t^3, t^2)$, for $m \in \mathbb{R}$. Show that $\lim_{t\to 0} f(\gamma(t)) = 1$.

Hence, determine whether $\lim_{(x,y)\to(0,0)} f(x, y)$ exists or not.

9. (a) Prove that for all $u > 0$, we have

$$\frac{1}{1+u^2} < \frac{\tan^{-1} u}{u} < 1.$$

(b) Using the result in (a), evaluate $\lim_{(x,y)\to(0,0)} \frac{\tan^{-1}(|x| + |y|)}{|x| + |y|}$.